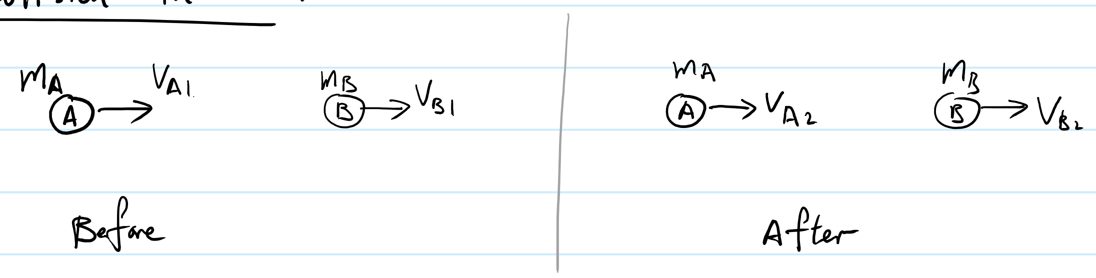
Lecture 7

Elastic collision in 1D.



Suppose m_A, m_B, V_{A1}, V_{B1} are given, we want to solve V_{A2}, V_{B2}

The solution is unique only if we impose two equations.

In 1D elastic collision, we have exactly 2 conservation law.

$$\begin{cases} \Delta E = 0 \\ \Delta \vec{p} = \vec{0} \end{cases}$$

$$\Delta E = 0 \Rightarrow \frac{1}{2} m_A V_{A1}^2 + \frac{1}{2} m_B V_{B1}^2 = \frac{1}{2} m_A V_{A2}^2 + \frac{1}{2} m_B V_{B2}^2$$

$$\Rightarrow m_A V_{A1}^2 + m_B V_{B1}^2 = m_A V_{A2}^2 + m_B V_{B2}^2 \quad \text{—— } ①$$

$$\Delta p = 0 \Rightarrow m_A V_{A1} + m_B V_{B1} = m_A V_{A2} + m_B V_{B2} \quad \text{—— } ②$$

from ①:

$$m_A V_{A1}^2 + m_B V_{B1}^2 = m_A V_{A2}^2 + m_B V_{B2}^2$$

$$m_A V_{A1}^2 - m_A V_{A2}^2 = -m_B V_{B1}^2 + m_B V_{B2}^2$$

$$\Rightarrow m_A (V_{A1}^2 - V_{A2}^2) = m_B (V_{B2}^2 - V_{B1}^2)$$

$$\Rightarrow \underbrace{m_A (V_{A1} - V_{A2})}(V_{A1} + V_{A2}) = \underbrace{m_B (V_{B2} - V_{B1})}(V_{B2} + V_{B1}) \quad \text{—— } ①'$$

from ②

$$m_A V_{A1} + m_B V_{B1} = m_A V_{A2} + m_B V_{B2}$$

$$\underbrace{m_A (V_{A1} - V_{A2})} = \underbrace{m_B (V_{B2} - V_{B1})}$$

putting the previous equation to ①'

$$V_{A1} + V_{A2} = V_{B2} + V_{B1}$$

$$\Rightarrow \quad V_{A1} - V_{B1} = -(V_{A2} - V_{B2}) \quad \text{———} \quad ③$$

or $$V_{A/B\,1} = -V_{A/B\,2}$$

(Velocity of A relative to B before collision) = − (Velocity of A relative to B after collision)

①, ② & ③ are true for 1D elastic collision.

② & ③ are linear. preferred!

① is quadratic.

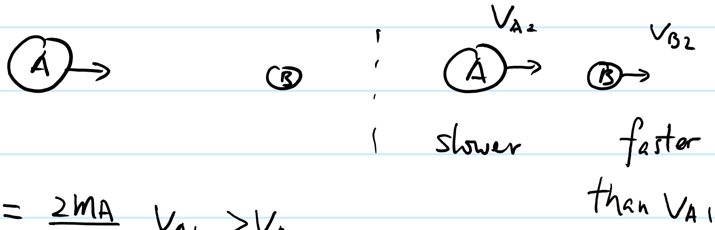
Using ② & ③ to solve for V_{A2} and V_{B2}

$$\begin{cases} V_{A2} = \dfrac{m_A - m_B}{m_A + m_B} V_{A1} + \dfrac{2m_B}{m_A + m_B} V_{B1} \\[2ex] V_{B2} = \dfrac{2m_A}{m_A + m_B} V_{A1} + \dfrac{m_B - m_A}{m_A + m_B} V_{B1} \end{cases}$$

Special cases: $V_{B1} = 0$ Stationary target.

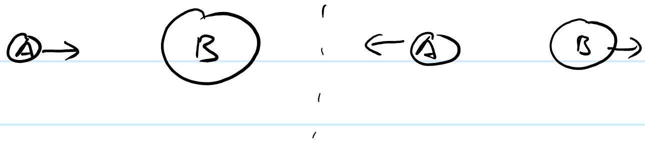
$$\begin{cases} V_{A2} = \dfrac{m_A - m_B}{m_A + m_B} V_{A1} \\[2ex] V_{B2} = \dfrac{2m_A}{m_A + m_B} V_{A1} \end{cases}$$

Case 1 $\quad m_A > m_B$



$$V_{A2} = \underbrace{\frac{m_A - m_B}{m_A + m_B}}_{\substack{>0 \\ \text{but } <1}} V_{A1} \quad , \quad V_{B2} = \underbrace{\frac{2m_A}{m_A + m_B}}_{>1} V_{A1} > V_{A1}$$

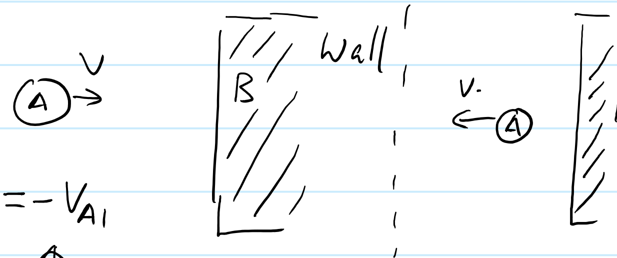
Case 2a $\quad m_A < m_B$



$$V_{A2} = \frac{m_A - m_B}{m_A + m_B} V_{A1} < 0$$

$$V_{B2} > 0$$

Case 2b $\quad m_B \to \infty$



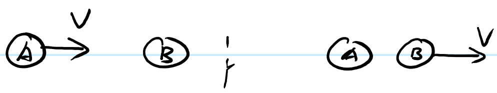
$$V_{A2} = \frac{m_A - m_B}{m_A + m_B} V_{A1} \to \frac{-m_B}{m_B} V_{A1} = -V_{A1}$$

bounce back with same speed.

$$V_{B2} = \frac{2m_A}{m_A + m_B} V_{A1} \to \frac{2m_A}{m_B} V_{A1} \to 0$$

Wall doesn't move.

Case 3 $\quad m_A = m_B$



exchange speed.

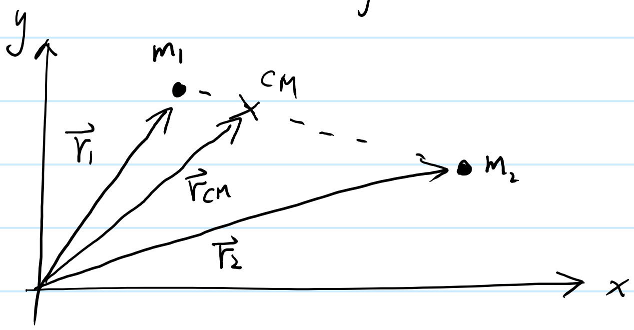
$$V_{A2} = \frac{m_A - m_B}{m_A + m_B} V_{A1} = 0$$

$$V_{B2} = \frac{2m_A}{m_A + m_B} V_{A1} = V_{A1}$$

Demo: Newton's cradle

Center of Mass (CM)

Definition: Center of mass of a system of particles can be considered as the <u>average position</u> of all the masses of the object.

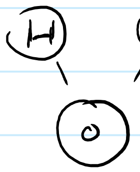


Position vector of CM for discrete particles.

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} \rightarrow \frac{\sum m_i \vec{r}_i}{\sum m_i} \longrightarrow \frac{\int dm \cdot \vec{r}}{m_{tot}}$$

$$\vec{r}_{cm} \begin{cases} x_{cm} = \dfrac{m_1x_1 + m_2x_2}{m_1+m_2} \\[2ex] y_{cm} = \dfrac{m_1y_1 + m_2y_2}{m_1+m_2} \\[2ex] z_{cm} = \dfrac{m_1z_1 + m_2z_2}{m_1+m_2} \end{cases}$$

<u>example</u>. H_2O



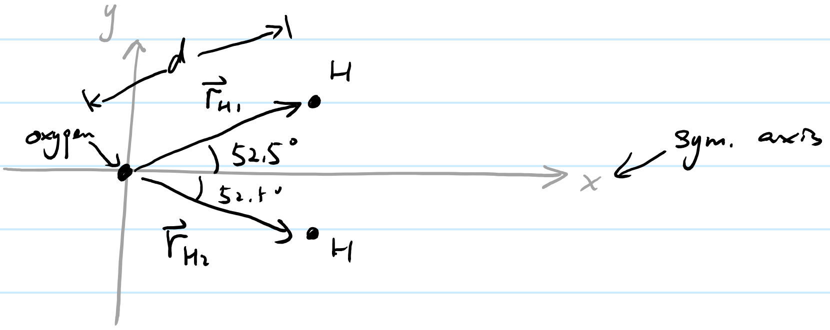
① Recognize the symmetry axis.

CM lies on sym. axis.

② choose a convenient coordinate system.

$m_H = 1u$

$m_O = 16u$



$$\vec{r}_{H_1} = d\cos(52.5^\circ)\,\hat{i} + d\sin(52.5^\circ)\,\hat{j}$$

$$\vec{r}_{H_2} = d\cos(52.5^\circ)\,\hat{i} - d\sin(52.5^\circ)\,\hat{j}$$

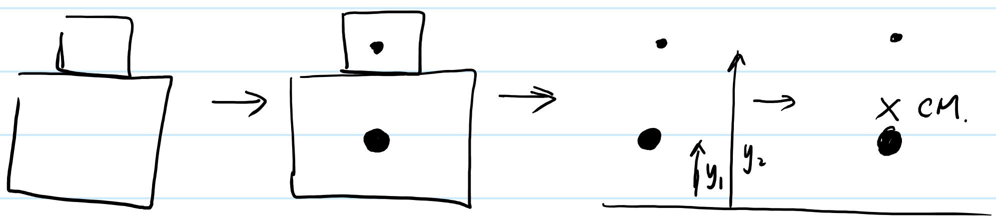
$$\vec{r}_O = \vec{0}$$

$$\Rightarrow \quad X_{CM} = \frac{1u \times d\cos(52.5^\circ) + 1u \times d\cos(52.5^\circ) + 0}{1u + 1u + 16u}$$

$$= 0.068\ d \qquad \text{where } d = 9.57 \times 10^{-11}\,\text{m}.$$

$$Y_{CM} = \frac{1u\ d\sin(52.5^\circ) - 1u\ d\sin(52.5^\circ) + 0}{18u} = 0$$

For composite objects, we take the CM of each sub-objects to represent the whole sub-object.



Newton's Law and CM

$$\vec{r}_{cm} = \frac{\sum m_i \vec{r}_i}{m_{tot}}$$

$$m_{tot}\,\vec{r}_{cm} = \sum m_i \vec{r}_i$$

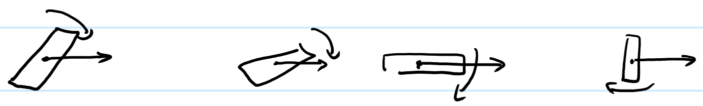
taking $\frac{d}{dt}$

$$\underline{m_{tot}\,\vec{V}_{cm}} = \sum m_i \vec{v}_i = \sum \vec{p}_i = \underline{\vec{P}_{tot}}.$$

It means total momentum of a system, such as a car, equals to the total mass of the system, m_{tot}, times the velocity of its CM.

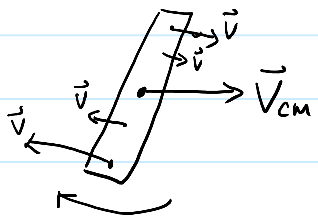
No need to keep track on all individual particles in the sys.

$$\vec{P}_{tot} = m_{tot}\vec{V}_{cm}$$



moving and rotating,

even each part of the object has a different velocity.



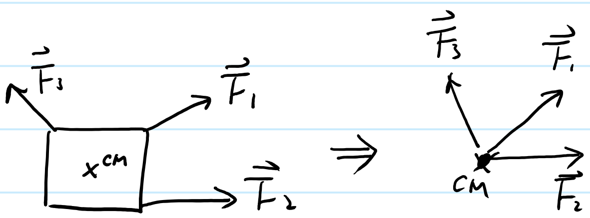
velocity of a special point on the object, CM.

But its momentum is still $m_{tot}\cdot\vec{V}_{CM}$

Let's take one more time derivative, we have

$$m_{tot}\frac{d\vec{V}_{cm}}{dt} = \frac{d}{dt}\vec{P}_{tot}$$

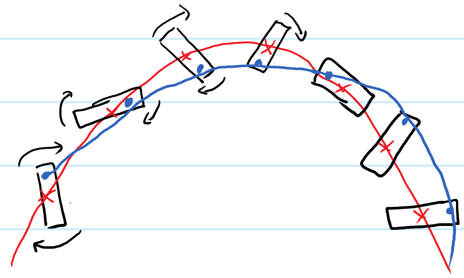
$$\Rightarrow m_{tot}\,\vec{a}_{CM} = \vec{F}_{ext}^{\,sys.}$$



It means the acceleration of the CM follows Newton's 2^{nd} Law as all the external forces are acting at the CM.

Again, not every point on the object would accelerate due to only the external forces, but the CM would.

<u>Example</u> projectile of a block with rotation.



CM: follows parabola motion

result of $F = mg = ma_{cm}$.

Blue pt.: does not follow parabola.

$\because$ motion of individual point is not determined by only the external forces, but also the internal ones.

Good news: No matter what shape the object has, the motion of its CM is determined by only the external force following Newton's Laws.

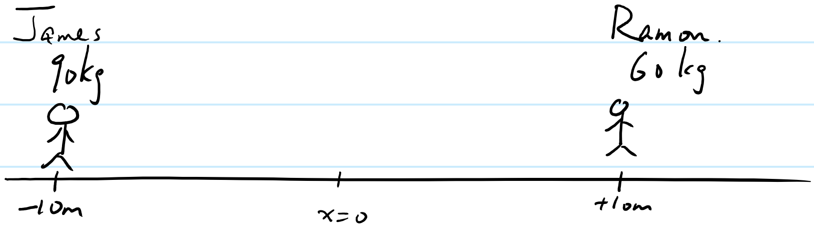
↳ It is why Free-body diagram works!

If $\vec{F}_{ext} = \vec{0}$, then $\vec{a}_{cm} = \vec{0}$.

i.e. for an isolated system (no external force), its CM must has a simple uniform motion $\begin{cases} \text{at rest} \\ \text{constant velocity.} \end{cases}$

but its shape can still change.

Example.



icy surface $\mu_s = \mu_k = 0$

(a) where is X_{cm}?

(b) If James pull Ramon, and they both move towards the center.

When James is at $x = -4m.$, where is Ramon?

(a) $$X_{cm} = \frac{90\cdot(-10m) + 60\cdot(10m)}{150} = \frac{-300}{150} = -2m.$$

(b) No matter who pull the rope, the rope will have the same tension on both ends, Therefore, they are both subjected to the same net force.

Consider they are in the same system,

$$\vec{F}_{ext} = \vec{0} \Rightarrow \vec{a}_{cm} = \vec{0}$$

$$\Rightarrow x_{cm} = -2m \text{ forever.}$$

When James is at $-4m$, Let x_R be position of Ramon,

$$-2 = X_{cm} = \frac{90\cdot(-4) + 60\cdot x_R}{150}$$

$$-300 = -360 + 60x_R$$

$$x_R = 1m. \#$$